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The operator formalism of classical statistical dynamics

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Abstract. The consistency of some assumptions made by Martin, Siggia and Rose in their operator formalism for the statistical dynamics of classical systems is demonstrated. This is achieved by the introduction of a simple representation for the operators involved.

1. Introduction

In a recent paper Martin *et al* (1973) have developed a new formalism for the discussion of the statistical dynamics of classical systems. They have pointed out the desirability of obtaining for classical systems a theory with the power and generality of the functional methods of Schwinger or the perturbation theory of Feynman which have proved so useful in quantum theory. The most serious attempts to develop such an approach for classical systems have been made in turbulence theory by Wyld, Kraichnan and Edwards (for references see Martin *et al* 1973). However, as Martin *et al* point out, the work of Wyld is correct only to fourth order in the anharmonicity because of an incorrect vertex renormalization. This flaw was recognized by Kraichnan who pointed out the necessity of introducing three vertex functions and, in unpublished work, derived the correct renormalized expansions. The formalism of Martin *et al* provides a much simpler derivation of these results. The theory of Edwards is of a rather different sort involving no vertex renormalization and is limited to turbulence maintained by stirring forces with delta function time correlations. The formalism of Martin *et al* suggests how this theory may be generalized as will be described later.

In the work of Martin *et al* a novel procedure is adopted which enables a Schwinger-type functional formalism to be developed and renormalized expansions obtained. This will now be briefly described. The basic dynamical variables of the system are denoted by $\psi(1)$ where 1 represents a time t_1 together with spatial coordinates $\boldsymbol{x}_1$ or other variables, either discrete or continuous, depending on the particular system under investigation. The quantity ψ is taken to satisfy an equation of motion of the form

$$\dot{\psi}(1) = U_1(1) + \int \mathrm{d}2\, U_2(1,2)\psi(2) + \int \mathrm{d}2 \int \mathrm{d}3\, U_3(1,2,3)\psi(2)\psi(3)$$

where U_2, U_3 contain delta functions in the time differences

$$U_2(1,2) \propto \delta(t_1 - t_2)$$
$$U_3(1,2,3) \propto \delta(t_1 - t_2)\delta(t_2 - t_3).$$

Examples of systems which permit of such a description are the damped anharmonic oscillator, a system of particles interacting through two-body forces, and a Navier-Stokes fluid.

A probability distribution is specified for the initial values of ψ and the problem is to determine the correlation functions of values of ψ at subsequent times. Martin *et al* point out that, whereas for a quantum mechanical system expectation values of products of the field operator ψ and its Hermitian conjugate $\psi^\dagger$ describe both correlations in the system *and* its response to external perturbations, for the classical system no counterpart of $\psi^\dagger$ appears and so the response functions do not enter automatically into the theory. This defect they remedy by introducing an operator $\hat{\psi}$ which satisfies the equal-time commutation relations

$$[\psi(\boldsymbol{x}, t), \hat{\psi}(\boldsymbol{x}', t)] = \delta(\boldsymbol{x}-\boldsymbol{x}')$$

together with an equation of motion and certain other conditions. This operator is interpreted as an excitation operator which causes small perturbations of ψ. The justification for this interpretation is not given in the paper and it is implied that it consists of verifying that the formal expressions presented for correlation and response functions in terms of ψ, $\hat{\psi}$ give the correct perturbation series for these quantities to the first few orders.

In this paper we should like to give a representation of the operator $\hat{\psi}$ in order to clarify the formalism and check its self-consistency. This is done by interpreting ψ and $\hat{\psi}$ as operators acting on functions of the initial values of ψ. We shall actually consider a more general equation of motion than the one written above.

2. Derivation of the operator formalism

The basic dynamical variables of the system are denoted by $\psi_n(t)$ where n is an index which can be either discrete or continuous. We shall here treat n as discrete to keep the notation simple without any loss of generality. The values of $\psi_n(t)$ at the initial instant $t = 0$ are denoted by ϕ_n. We assume that the initial data have a probability distribution specified by a density function $\rho(\phi)$ where ϕ denotes the set $(\phi_1, \phi_2, \ldots)$ of all the variables. The equation of motion is taken to be

$$\dot{\psi}_n(t) = \Omega_n(\psi(t), t) \tag{1}$$

where the Ω_n are given functions of $\psi(t)$ and t.

We now consider a set of real functions of the initial values $\Phi(\phi)$, $\Psi(\phi)$, etc (denoted by Greek capital letters) which satisfy the condition

$$\int \mathrm{d}\phi \rho(\phi)\Phi^2(\phi) < \infty$$

so that they can be regarded as elements of a real Hilbert space $\mathscr{H}$ in which is defined the scalar product

$$(\Phi, \Psi) = (\Psi, \Phi) = \int \mathrm{d}\phi \rho(\phi)\Psi(\phi)\Phi(\phi).$$

If we assume that the equation of motion (1) determines $\psi(t)$ uniquely in terms of ϕ, then the quantity $\Phi(\psi(t))$ can be regarded as a function of ϕ and will be written $\Phi_t(\phi)$. This can be taken as arising from the action on $\Phi(\phi)$ of a linear operator $E(t)$,

$$\Phi_t(\phi) = E(t)\Phi(\phi). \tag{2}$$

An obvious property of E is the multiplicative one,

$$E(t)\Phi(\phi)\Psi(\phi) = \{E(t)\Phi(\phi)\}\{E(t)\Psi(\phi)\}.$$

An equation of motion for the operator $E(t)$ may be easily derived from (1) and (2):

$$\dot{E}(t)\Phi(\phi) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\psi(t)) = \dot{\psi}_n(t)\Phi_{,n}(\psi(t))$$

where the comma notation denotes differentiation, and a summation over repeated indices is implied. Thus

$$\dot{E}(t)\Phi(\phi) = \Omega_n(\psi(t), t)\Phi_{,n}(\psi(t)) = E(t)\Omega_n(\phi, t)\Phi_{,n}(\phi).$$

Hence

$$\dot{E}(t) = E(t)\mathscr{L}(t) \tag{3}$$

where $\mathscr{L}(t)$ denotes the operator $\Omega_n(\phi, t)(\partial/\partial\phi_n)$. Since $E(0) = 1$ we may write the integral equation

$$E(t) = 1 + \int_0^t \mathrm{d}\tau E(\tau)\mathscr{L}(\tau).$$

An equation for the inverse operator $E^{-1}(t)$ may be obtained by differentiating the identity

$$E(t)E^{-1}(t) = 1.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{-1}(t) = -E^{-1}(t)\dot{E}(t)E^{-1}(t) = -\mathscr{L}(t)E^{-1}(t). \tag{4}$$

For future reference it is convenient to note that the operator $X(t, t')$ defined as $E^{-1}(t')E(t)$ satisfies the equation

$$X(t, t') = 1 + \int_{t'}^t \mathrm{d}\tau\, X(\tau, t')\mathscr{L}(\tau). \tag{5}$$

We now define operators acting in $\mathscr{H}$ in the following way:

$$\begin{aligned} A_n(t) &= E(t)\phi_n E^{-1}(t) \\ B_n(t) &= E(t)\frac{\partial}{\partial\phi_n}E^{-1}(t) \end{aligned} \tag{6}$$

where ϕ_n is to be regarded as the operator corresponding to multiplication by ϕ_n. It is seen immediately that $A_n(t)$ corresponds to multiplication by $\psi_n(t)$ since we have

$$A_n(t)\Phi(\phi) = E(t)\phi_n E^{-1}(t)\Phi(\phi) = \{E(t)\phi_n\}\{E(t)E^{-1}(t)\Phi(\phi)\} = \psi_n(t)\Phi(\phi)$$

and we shall henceforth denote this operator by the same symbol $\psi_n(t)$.

The equal-time commutation relations follow from the definitions

$$[\psi_n(t), \psi_m(t)] = [B_n(t), B_m(t)] = 0$$
$$[B_n(t), \psi_m(t)] = \delta_{nm}$$

and we also have

$$[\psi_n(t), \psi_m(t')] = 0.$$

The equations of motion follow from those of $E(t)$ and $E^{-1}(t)$:

$$\dot{\psi}_n(t) = E(t)[\mathscr{L}(t), \phi_n]E^{-1}(t) = [L(t), \psi_n(t)]$$

and

$$\dot{B}_n(t) = [L(t), B_n(t)]$$

where $L(t)$ denotes the operator $E(t)\mathscr{L}(t)E^{-1}(t)$. We can write

$$L(t) = \Omega_n(\psi(t), t)B_n(t)$$

and the equations of motion are

$$\dot{\psi}_n(t) = \Omega_n(\psi(t), t)$$
$$\dot{B}_n(t) = -\Omega_{m,n}(\psi(t), t)B_m(t).$$

(From the equation of motion for $B_n(t)$ it can be shown that this operator may be equated to $\partial/\partial\psi_n(t)$, where it is understood that the function of ϕ on which it acts is rewritten as a function of $\psi(t)$.)

It is convenient at this stage to introduce the adjoint $B_n^\dagger$ of the operator B_n in the usual way:

$$(B_n^\dagger\Psi, \Phi) = (\Psi, B_n\Phi).$$

The operator $B_n^\dagger$ coincides with the operator $\hat{\psi}_n$ of Martin *et al* as will soon become apparent. The operator $\psi_n(t)$ is clearly self-adjoint. Taking the adjoint of the commutation relations gives

$$[\hat{\psi}_n(t), \hat{\psi}_m(t)] = 0$$
$$[\psi_n(t), \hat{\psi}_m(t)] = \delta_{nm}$$

while the adjoint of the equations of motion gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\psi}_n(t) = -\hat{\psi}_m(t)\Omega_{m,n}(\psi(t), t).$$

If the equation of motion of ψ is of the form

$$\dot{\psi}(1) = U_1(1) + \int d2\, U_2(1,2)\psi(2) + \int d2 \int d3\, U_3(1,2,3)\psi(2)\psi(3)$$

then we have

$$\dot{\hat{\psi}}(1) = -\int d2\, U_2(2,1)\hat{\psi}(2) - 2\int d2 \int d3\, U_3(2,3,1)\hat{\psi}(2)\psi(3),$$

which agrees with the equation given by Martin *et al.*

3. Formal expressions for correlation and response functions

To obtain formal expressions for correlation and response functions in terms of the operators ψ, $\hat{\psi}$ we now introduce the unit function Φ_0 which plays the role of the vacuum state

$$\Phi_0(\phi) \equiv 1.$$

Correlation functions can then be written in the form of vacuum expectation values, for example

$$\mathscr{E}\{\psi_n(t)\psi_m(t')\} = (\Phi_0, \psi_n(t)\psi_m(t')\Phi_0)$$

where $\mathscr{E}$ denotes expectation value. Using the commutativity of the ψ this can be re-written as

$$(\Phi_0, T\{\psi_n(t)\psi_m(t')\}\Phi_0).$$

Here T is the usual chronological ordering which arranges operators in order of increasing time from right to left, so that for example

$$T\{A(1)B(2)C(3)\} = \theta(1,2)\theta(2,3)A(1)B(2)C(3) + \theta(2,1)\theta(1,3)B(2)A(1)C(3) + \dots$$

where $\theta(1,2)$ denotes

$$\theta(t_1 - t_2) = \begin{cases} 1 & t_1 > t_2 \\ 0 & t_1 < t_2. \end{cases}$$

It is understood that if a ψ and a $\hat{\psi}$ operator in a T product have the same time then $\hat{\psi}$ stands to the left of ψ.

To investigate the response functions of the system we now suppose that $\Omega_n(\psi(t), t)$ involves an external force $f_n(t)$ additively, so that

$$\Omega_n(\psi(t), t) = f_n(t) + \Lambda_n(\psi(t), t)$$

where Λ does not explicitly depend on f.

We first evaluate the functional derivative $\delta E(t)/\delta f_m(t')$. Differentiating the identity

$$E(t) = 1 + \int_0^t d\tau\, E(\tau)\mathscr{L}(\tau)$$

gives

$$\begin{aligned}\frac{\delta E(t)}{\delta f_m(t')} &= \int_0^t \mathrm{d}\tau E(\tau)\frac{\delta \mathscr{L}(\tau)}{\delta f_m(t')} + \int_0^t \mathrm{d}\tau \frac{\delta E(\tau)}{\delta f_m(t')}\mathscr{L}(\tau) \\ &= \int_0^t \mathrm{d}\tau E(\tau)\delta(\tau - t')\frac{\partial}{\partial \phi_m} + \int_0^t \mathrm{d}\tau \frac{\delta E(\tau)}{\delta f_m(t')}\mathscr{L}(\tau) \\ &= \theta(t-t')E(t')\frac{\partial}{\partial \phi_m} + \int_0^t \mathrm{d}\tau \frac{\delta E(\tau)}{\delta f_m(t')}\mathscr{L}(\tau).\end{aligned}$$

Clearly we have

$$\frac{\delta E(t)}{\delta f_m(t')} = 0 \qquad \text{for } t < t'$$

which is simply an expression of causality, while for $t > t'$ we may write

$$\frac{\delta E(t)}{\delta f_m(t')} = E(t')\frac{\partial}{\partial \phi_m}X_m(t, t')$$

where X_m satisfies the equation

$$X_m(t, t') = 1 + \int_{t'}^t \mathrm{d}\tau X_m(\tau, t')\mathscr{L}(\tau).$$

Comparison of this with (5) shows that $X_m(t, t')$ is in fact independent of m and is equal to $E^{-1}(t')E(t)$. We can therefore write

$$\frac{\delta E(t)}{\delta f_m(t')} = \theta(t-t')E(t')\frac{\partial}{\partial \phi_m}E^{-1}(t')E(t) = \theta(t-t')B_m(t')E(t).$$

Similarly

$$\frac{\delta E^{-1}(t)}{\delta f_m(t')} = -\theta(t-t')E^{-1}(t)B_m(t').$$

From these results it follows that

$$\begin{aligned}\frac{\delta \psi_n(t)}{\delta f_m(t')} &= \frac{\delta}{\delta f_m(t')}\{E(t)\phi_n E^{-1}(t)\} \\ &= \theta(t-t')\{B_m(t')E(t)\phi_n E^{-1}(t) - E(t)\phi_n E^{-1}(t)B_m(t')\} \\ &= \theta(t-t')[B_m(t'), \psi_n(t)]\end{aligned}$$

and taking the adjoint,

$$\frac{\delta \psi_n(t)}{\delta f_m(t')} = \theta(t-t')[\psi_n(t), \hat{\psi}_m(t')]; \tag{7}$$

similarly

$$\frac{\delta \hat{\psi}_n(t)}{\delta f_m(t')} = \theta(t-t')[\hat{\psi}_n(t), \hat{\psi}_m(t')]. \tag{8}$$

Using (7) and (8) we can write operator expressions for response functions, for example

$$\mathscr{E}\left\{\frac{\delta\psi_n(t)}{\delta f_m(t')}\right\} = (\Phi_0, [\psi_n(t), \hat{\psi}_m(t')]\Phi_0)\theta(t-t')$$

and since $(\Phi_0, \hat{\psi}\Phi) = 0$, this can be rewritten as a T product,

$$(\Phi_0, T\{\psi_n(t)\hat{\psi}_m(t')\}\Phi_0).$$

Similarly we have

$$\mathscr{E}\left\{\frac{\delta}{\delta f_p(t'')}\psi_n(t)\psi_m(t')\right\} = (\Phi_0, T\{\psi_n(t)\psi_m(t')\hat{\psi}_p(t'')\}\Phi_0).$$

This completes the demonstration of the consistency of the operator formalism.

4. The functional formalism

Taking these formal expressions for correlation and response functions together with the equations of motion of the operators involved, it is a simple matter to generate the infinite hierarchy of equations satisfied by these quantities. It is more convenient however to introduce a generating functional in the manner of Schwinger. The idea of replacing the infinite hierarchy by a single functional differential equation goes back to the work of Hopf in turbulence theory and has been developed in this field by Edwards (1964), Herring (1966) and others. (These later authors formulate their theories in terms of the probability density function rather than the characteristic functional which is the generating functional of Hopf. Since these quantities are just functional Fourier transforms of each other it is easy to express this later work in Hopf's formalism.) This work has suffered from some basic limitations however. Firstly the generating functionals used have not borne any simple relation to the response functions which therefore do not appear naturally in the theory. In addition they have usually been of the single-time variety so that non-simultaneous correlations are difficult to handle. Also it has not been possible to deal in a general manner with the situation in which a random stirring force acts on the system. This is necessary for a discussion of statistically stationary states of dissipative systems, such as stationary turbulence of a Navier–Stokes fluid. The only sort of random force which it has been found possible to treat in these theories is one having delta function time correlations, since only in this case can a closed equation for the characteristic functional be obtained.

The new operator formalism enables these problems to be solved in a simple way. We now take the equations of motion as

$$\dot{\psi}_n(t) = f_n(t) + \Lambda_n(\psi(t), t)$$
$$\dot{\hat{\psi}}_n(t) = -\hat{\psi}_m(t)\Lambda_{m,n}(\psi(t), t)$$

where f is a Gaussian random function of zero mean with a correlation function given by

$$\langle f_n(t) f_m(t')\rangle = R_{nm}(t, t').$$

Furthermore it is assumed that f is statistically independent of the initial values of ψ. Expectation values with respect to the distribution of f are denoted by angular brackets as above.

The generating functional is given by

$$Z[\xi,\eta] = \left(\Phi_0, \left\langle T\exp\int dt\{\xi_n(t)\psi_n(t)+\eta_n(t)\hat{\psi}_n(t)\}\right\rangle\Phi_0\right)$$

where ξ and η are suitably well behaved test functions. We shall adopt a shorthand notation in which $\xi(1)$ denotes $\xi_{n_1}(t_1)$ and $\int d1$ denotes $\Sigma_{n_1}\int dt_1$ etc. Forming the functional derivative $\delta Z/\delta\xi(1)$, differentiating with respect to t_1 and using the equation of motion for ψ, we obtain

$$\frac{\partial}{\partial t_1}\frac{\delta Z}{\delta\xi(1)} = \eta(1)Z+\Lambda_{n_1}\left(\frac{\delta}{\delta\xi(t_1)}, t_1\right)Z$$
$$+\sum_p\frac{1}{(p-1)!}\int d2\ldots\int dp(\Phi_0, \langle T\{f(1)c(2)\ldots c(p)\}\rangle\Phi_0)$$

where we have written $c(2)$ for $\xi(2)\psi(2)+\eta(2)\hat{\psi}(2)$, ie $c_n(t) = \xi_n(t)\psi_n(t)+\eta_n(t)\hat{\psi}_n(t)$ (no summation over n).

The first term on the right-hand side arises from the differentiation of the step functions implicit in the time ordering while the remaining terms come from $\dot{\psi}(1)$. The only complication here which does not appear in the usual Schwinger theory is the last term on the right which has not yet been expressed in terms of Z. This however may be accomplished by making use of a theorem due to Novikov (1965): that if $A[f]$ is any functional of the Gaussian random function f then

$$\langle f(1)A[f]\rangle = \int d2R(1,2)\left\langle\frac{\delta A[f]}{\delta f(2)}\right\rangle.$$

It may be verified that the result is also true if A is an operator functionally dependent on f. Applying this result to the terms involving f enables them to be reduced to T products of the operators ψ, $\hat{\psi}$. For example,

$$\int d2\int d3\int d4(\Phi_0, \langle T\{f(1)c(2)c(3)c(4)\}\rangle\Phi_0)$$
$$= 3!\int d2\int d3\int d4\theta(2,3)\theta(3,4)(\Phi_0, \langle f(1)c(2)c(3)c(4)\rangle\Phi_0)$$

(since f commutes with c)

$$= 3!\int d2\ldots\int d5\theta(2,3)\theta(3,4)R(1,5)\left(\Phi_0, \left\langle\frac{\delta}{\delta f(5)}c(2)c(3)c(4)\right\rangle\Phi_0\right)$$

by use of Novikov's theorem

$$= 3!\int d2\ldots\int d5\theta(2,3)\theta(3,4)R(1,5)$$
$$\times(\Phi_0, \langle\theta(2,5)[c(2),\hat{\psi}(5)]c(3)c(4)+\theta(3,5)c(2)[c(3),\hat{\psi}(5)]c(4)$$
$$+\theta(4,5)c(2)c(3)[c(4),\hat{\psi}(5)]\rangle\Phi_0)$$

because of (7) and (8).

This can be rewritten in the form

$$\int d2\ldots\int d5R(1,5)(\Phi_0, \langle T\{\hat{\psi}(5)c(2)c(3)c(4)\}\rangle\Phi_0).$$

Combining all such terms gives finally

$$\int \mathrm{d}2 R(1,2) \frac{\delta Z}{\delta \eta(2)}.$$

Dealing in the same way with $(\partial/\partial t_1)(\delta Z/\delta \eta(1))$ we obtain the functional differential equations

$$\frac{\partial}{\partial t} \frac{\delta Z}{\delta \xi_n(t)} = \eta_n(t) Z + \Lambda_n\left(\frac{\delta}{\delta \xi(t)}, t\right) Z + \int \mathrm{d}t' R_{nm}(t, t') \frac{\delta Z}{\delta \eta_m(t')} \tag{9}$$

$$\frac{\partial}{\partial t} \frac{\delta Z}{\delta \eta_n(t)} = -\xi_n(t) Z - \frac{\delta}{\delta \eta_m(t)} \Lambda_{m,n}\left(\frac{\delta}{\delta \xi(t)}, t\right) Z. \tag{10}$$

For the equations of motion considered by Martin *et al* these become

$$\frac{\partial}{\partial t_1} \frac{\delta Z}{\delta \xi(1)} = \eta(1) Z + U_1(1) Z + \int \mathrm{d}2\, U_2(1,2) \frac{\delta Z}{\delta \xi(2)} + \int \mathrm{d}2 \int \mathrm{d}3\, U_3(1,2,3) \frac{\delta^2 Z}{\delta \xi(2) \delta \xi(3)} + \int \mathrm{d}2\, R(1,2) \frac{\delta Z}{\delta \eta(2)}$$

$$\frac{\partial}{\partial t_1} \frac{\delta Z}{\delta \eta(1)} = -\xi(1) Z - \int \mathrm{d}2\, U_2(2,1) \frac{\delta Z}{\delta \eta(2)} - 2 \int \mathrm{d}2 \int \mathrm{d}3\, U_3(2,3,1) \frac{\delta^2 Z}{\delta \eta(2) \delta \xi(3)}.$$

These equations may be used as a convenient basis from which to develop the renormalized expansions as described by Martin *et al.* They also reveal the interesting fact that, as far as the calculation of correlation and response functions is concerned, nothing is changed if in the equations of motion for ψ we replace $f(1)$ by $\int \mathrm{d}2 R(1,2) \hat{\psi}(2)$, provided that $R(1,2)$ contains a delta function $\delta(t_1 - t_2)$. (It is stated by Martin *et al* that this result is valid for general R but our argument does not suggest this.)

The equations (9) and (10) may be used to derive another equation for Z which characterizes a statistically stationary state (eg stationary turbulence).. For a stationary situation it is apparent that Z will be unchanged if the test functions ξ, η are replaced by the 'time-shifted' test functions defined by

$$\xi^{(\tau)}(\boldsymbol{x}, t) = \xi(\boldsymbol{x}, t-\tau)$$

$$\eta^{(\tau)}(\boldsymbol{x}, t) = \eta(\boldsymbol{x}, t-\tau).$$

This then shows that

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau} Z[\xi^{(\tau)}, \eta^{(\tau)}]\right]_{\tau=0} = 0$$

which may be expressed in the form

$$\int \mathrm{d}1 \left\{ \xi(1) \frac{\partial}{\partial t_1} \frac{\delta}{\delta \xi(1)} + \eta(1) \frac{\partial}{\partial t_1} \frac{\delta}{\delta \eta(1)} \right\} Z = 0.$$

Substituting from (9) and (10) gives

$$HZ = 0 \tag{11}$$

where H is the functional differential operator

$$\int \mathrm{d}t\xi_n(t)\Lambda_n\left(\frac{\delta}{\delta\xi(t)}, t\right) + \int \mathrm{d}t \int \mathrm{d}t'\xi_n(t)R_{nm}(t, t')\frac{\delta}{\delta\eta_m(t')} - \int \mathrm{d}t\eta_n(t)\frac{\delta}{\delta\eta_m(t)}\Lambda_{m,n}\left(\frac{\delta}{\delta\xi(t)}, t\right).$$

For the equations of motion of Martin *et al*, H is given by

$$\int \mathrm{d}1\xi(1)U_1(1) + \int \mathrm{d}1 \int \mathrm{d}2U_2(1,2)\left(\xi(1)\frac{\delta}{\delta\xi(2)} - \eta(2)\frac{\delta}{\delta\eta(1)}\right) + \int \mathrm{d}1 \int \mathrm{d}2R(1,2)\xi(1)\frac{\delta}{\delta\eta(2)}$$
$$+ \int \mathrm{d}1 \int \mathrm{d}2 \int \mathrm{d}3U_3(1,2,3)\left(\xi(1)\frac{\delta}{\delta\xi(2)} - 2\eta(2)\frac{\delta}{\delta\eta(1)}\right)\frac{\delta}{\delta\xi(3)}.$$

This equation seems to provide what Edwards (1964) refers to as the Lagrangian description of stationary turbulence (although this terminology is perhaps misleading), which he attempted to formulate in terms of the characteristic functional or, equivalently, in terms of the probability density. However, the derivation of such a closed equation in terms of the characteristic functional appears to be impossible. The equation (11) does not have a unique solution but it is possible to derive a series solution which automatically satisfies the more basic equations (9) and (10). Such methods have been investigated in the context of the quantum many-body problem by Edwards and Sherrington (1967) and it seems likely that an analogous theory could be based on (11) for problems such as stationary turbulence. Such a theory should not be confused with the single-time functional formalism for turbulence given by Edwards (1964). Incidentally it is apparent that the basic equation of the Edwards approach to the quantum many-body problem may be derived more easily by a method like the one given above: we simply take the Schwinger equations for the system and impose on them the condition that the time-shift operation should leave the generating functional unchanged.

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